Calculus II - Day 6

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Alternating Series, Absolute v. Conditional Convergence

Goals for today:

- find a criterion that guarantees alternating series converge
- estimate convergent alternating series and bound the error
- distinguish between absolute and conditional convergence of alternating series

Example (from last week): Does this series converge or diverge?

$$\sum_{k=1}^{\infty} \frac{2k^2+3k}{\sqrt{k^5+5}} \xrightarrow{\text{limit comparison test}} \sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^5}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \quad \textit{(diverges: p-series with } p = \frac{1}{2}\textit{)}$$

$$\frac{\lim_{k\to\infty}\left(\frac{2k^2+3k}{\sqrt{k^5+5}}\right)}{\frac{1}{\sqrt{k}}} = \lim_{k\to\infty}\left(\frac{(2k^2+3k)\sqrt{k}}{\sqrt{k^5+5}}\right) = \lim_{k\to\infty}\left(\frac{2k^{2.5}+3k^{1.5}}{\sqrt{k^5+5}}\right) = \frac{2}{1}$$

Since $\sum \frac{1}{\sqrt{k}}$ diverges, so does our series.

What if the signs in a series alternate?
Example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}$$

This is called the *Alternating Harmonic Series.*

We know the Harmonic Series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$ *(diverges)*, but the alternating signs change this behavior:

$$S_1 = 1$$

$$S_2 = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 0.8\overline{3}$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12} = 0.58\overline{3}$$

$$S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.78\overline{3}$$

$$S_6 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} = 0.61\overline{6}$$

The Alternating Series Test:

Let $\sum(-1)^k a_k$ or $\sum(-1)^{k+1} a_k$ be an alternating series (so a_k is positive for every k). If:

1) The terms of the series are non-increasing in absolute value

$$(0 < a_{k+1} \leq a_k),$$

2) $\lim_{k\to\infty} a_k = 0$,

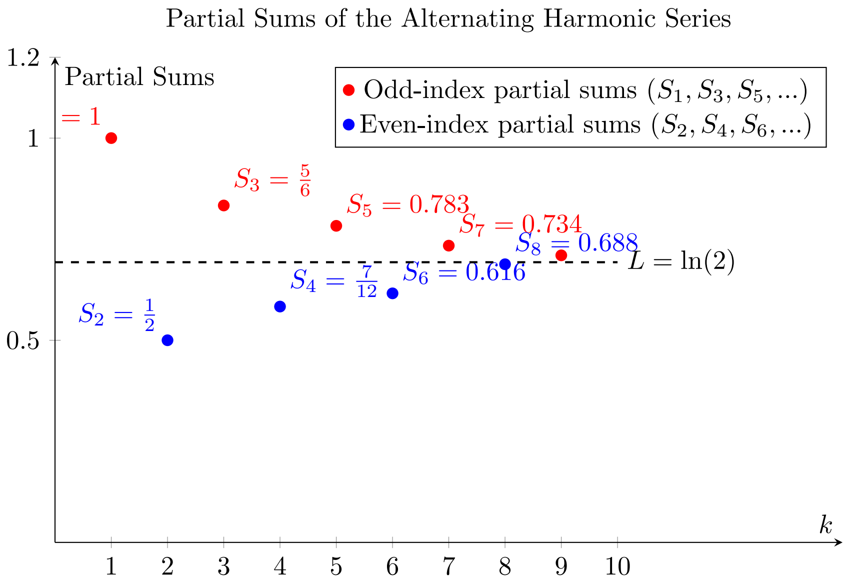
then the series converges.
Example: AHS (Alternating Harmonic Series):

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

is an alternating series.
Here, $a_k = \frac{1}{k}$.
Observe: $0 < \frac{1}{k+1} \leq \frac{1}{k}$ (terms are decreasing), and

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

So the AHS converges by the AST (Alternating Series Test).



Alternating Series Test

Alternating Series Test: similar to the Divergence Test

In general, the DT says that if $a_k \not\to 0$, $\sum a_k$ diverges. Usually, it's not true that $a_k \to 0$ is enough to say the series converges, but it *is* enough if the series is alternating.

Example:

$$-1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \cdots$$

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}}$$

This series converges by the AST:

- Alternating ✓
- $a_k = \frac{1}{\sqrt{k}}$: decreasing, goes to 0 ✓

Example:

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k}$$

- Alternating ✓
- Terms are decreasing in absolute value ✓
- But, $\lim_{k\to\infty} \frac{k+1}{k} = 1 \neq 0$ ×

Diverges by the Divergence Test.

Alternating series converge "quickly"

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

We can estimate S by looking at a partial sum:

$$S_N = \sum_{k=1}^{N} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots \pm a_N \quad \xleftarrow{\textit{+ if N is odd, - if N is even}}$$

Hopefully, if N is large, $S_N \approx S$

Q: How close?

Let $R_N = S - S_N$ be the Nth remainder.

To estimate S using S_N, we want $|R_N|$ to be small.

For alternating series, we can bound this quantity:

Theorem: (Remainders of Alternating Series)

Let $\sum (-1)^k a_k$ be a convergent alternating series with terms nonincreasing in absolute value. Let $R_N = S - S_N$ be the Nth remainder. Then

$$|R_N| \le a_{N+1}$$

Example: Approximate $S = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ using the 9th partial sum S_9

$$S_9 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9}$$

How close is this to the actual value of S?

By the theorem:

$$|R_9| = |S - S_9| \le a_{10} = \frac{1}{10}$$

Turns out:

$$S_9 = 0.74563\ldots \quad \text{and} \quad S = \ln(2) = 0.69314718\ldots$$

Example: Consider the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

This converges by the Alternating Series Test (AST). How many terms are needed to estimate the sum with an absolute error $|R_N| < 0.001$?

We want to find the smallest N such that

$$|R_N| \le a_{N+1} < 0.001$$

Here, $a_k = \frac{1}{k!}$, so we need

$$\frac{1}{(N+1)!} < \frac{1}{1000}$$

We want the smallest N such that:

$$\frac{1}{(N+1)!} < \frac{1}{1000} \Leftrightarrow (N+1)! > 1000$$

Calculating factorials:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720, \quad 7! = 7 \times 720 = 5040$$

Therefore, take $N + 1 = 7 \rightarrow N = 6$.

So, $S_6 = \sum_{k=1}^{6} \frac{(-1)^{k+1}}{k!}$ is within 0.001 of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} = 1 - \frac{1}{e}$.

The professor remarks that it's fascinating how e and π show up in places they seemingly have 'no business being in,' and later in the course, we'll explore why that is.

Alternating versus "traditional" harmonic series:

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} \quad \text{converges} \quad \Rightarrow \quad \sum_{k=1}^{\infty}\frac{1}{k} = \infty \textit{ (diverges)}$$

Definition: Let $\sum a_k$ be a series.

1) If $\sum |a_k|$ converges, we say $\sum a_k$ converges absolutely.

2) If $\sum |a_k|$ diverges, but $\sum a_k$ converges, we say $\sum a_k$ converges conditionally.

The AHS converges conditionally:

1) $\sum(-1)^{k+1}\frac{1}{k}$ converges, and

2) $\sum |(-1)^{k+1}\frac{1}{k}| = \sum \frac{1}{k}$ *diverges*

However,

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2}$$

converges absolutely:

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{1}{k^2}\right| = \sum_{k=1}^{\infty}\frac{1}{k^2} \quad \text{converges} \quad \textit{(p-series with p=2)}$$

Note: If $\sum a_k$ is a series where all terms are positive, it's impossible to converge conditionally. , because $\sum |a_k| = \sum a_k$

$\Rightarrow$ must either diverge or converge absolutely.

Theorem: If $\sum |a_k|$ converges, then $\sum a_k$ converges. (absolute convergence implies regular convergence)

Example: $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ — converge or diverge?
This series oscillates but isn't alternating.
To show this series converges, show it converges absolutely:

$$\sum_{k=1}^{\infty}\left|\frac{\sin(k)}{k^2}\right|$$

...to be continued...